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Matrices: Peculiar Determinant Property

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ABSTRACT

This paper provides a review of the theory surrounding matrices and determinants. While modern mathematics treats matrices and determinants as interconnected concepts, historically, determinants were recognized long before matrices were formally defined. The term "determinant" emerged in mathematical discourse over two centuries prior to the formal introduction of matrices. In this paper, we present a noteworthy property of square matrices: the divisibility of determinants. We show that for each row, where each row represents one number, are all divisible by some number d, then, in turn, the determinant of the matrix will also be divisible by the same number d. The findings of this study are of great importance as matrices with special additive determinant properties can be used for graph applications together with network analysis.

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1. Introduction

Matrices and determinants represent a captivating aspect of mathematics, appealing not only to those who study it but also to educators who aim to convey these concepts effectively. Throughout history, numerous mathematicians have left their mark on the development of matrices and determinants (Burton, 2010; Eves, 1990). The modern theory of determinants was significantly shaped by the insights of German mathematicians Karl Theodor Wilhelm Weierstrass and Leopold Kronecker, whose lectures laid the groundwork for future advancements (Kronecker, 1903). A novel method emerged for computing the determinant of matrices, presenting a fresh perspective on this fundamental mathematical concept (Rezaifar & Rezaee, 2007). Early pioneers associated determinants with polynomials, defining the term independently of the existence of a square matrix. Square matrices, encompassing a

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diverse range including Arrowhead matrix, Hadamard matrix, Sylvester matrix, Walsh matrix, Bézout matrix, Hessian matrix, Symplectic matrix, Bernoulli matrix, Hourglass matrix, Adjacency matrix, Edmonds matrix, Hat matrix, and Supnick matrix, possess unique characteristics that make them particularly appealing. This fascination often overshadows that of non-square matrices, as emphasized in (Babarinsa & Kamarulhaili, 2018).

A matrix is an arrangement of numbers into rows and columns. More specifically, a square matrix is a matrix with the same number of rows and columns. Matrices are fundamental in the real world. People from different careers and educational backgrounds use matrices throughout their work. For example, geologists use them to do seismic surveys, physicists use them to study quantum mechanics, and people who work with computers use them to project a 3-D image into a 2-D screen. This work is focused on a particular property of square matrices called divisibility of determinants.

The divisibility of determinants within square matrices serves as a captivating area of study, offering profound insights into the underlying structures and properties of these mathematical constructs. By exploring the divisibility of determinants, researchers delve into the intricate interplay between the elements within matrices, unraveling patterns and relationships that underpin their mathematical behavior. This pursuit extends beyond mere theoretical conjecture, finding practical relevance in various fields where matrices are indispensable tools for problem-solving and analysis. Understanding the divisibility of determinants empowers professionals to optimize their use of matrices in diverse applications, enhancing their efficacy in tasks ranging from data analysis to algorithm design. Some significant advancements have been made regarding the divisibility among determinants of power matrices. Also, there are studies related to this topic (Chen & Hong, 2020; Feng, Hong & Zhao, 2009; Li & Tan, 2011), have considerably contributed to the literature. This paper aims to explore a specific attribute of square matrices concerning the divisibility of determinants.

Moreover, the study of the divisibility of determinants within square matrices represents a testament to the enduring relevance and versatility of mathematical concepts across different domains of knowledge. As scholars probe deeper into this phenomenon, they uncover connections that transcend disciplinary boundaries, shedding light on the underlying principles governing complex systems and phenomena.

2. Methodology

2.1. Example and Theorems

Consider the square matrix A of order 2:

$$A = \begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix}$$

The determinant of matrix A is 45, which is divisible by 9 since -45=9(-5). In the above matrix, observe that the rows of matrix concatenated together gives us number 27, and 72 which are both divisible by 9.

In the theorems mentioned below, it is asserted that the determinant of matrix A cannot be zero. This is because a determinant of zero indicates that the matrix lacks an inverse. Further discussion on this topic will follow later.

Theorem A. Let $\mathbf{A} = \begin{bmatrix} \mathbf{a_{11}} & \mathbf{a_{12}} \\ \mathbf{a_{21}} & \mathbf{a_{22}} \end{bmatrix}$ and $\mathbf{det}\,(\boldsymbol{A}) \neq \mathbf{0}$. If $\mathbf{10a_{11}} + \mathbf{a_{12}}$, and $\mathbf{10a_{21}} + \mathbf{a_{22}}$ are divisible by the same nonzero number $\boldsymbol{d}$, then the determinant of $\boldsymbol{A}$ is also divisible by $\boldsymbol{d}$.

Proof of theorem A: Since $\mathbf{10}\boldsymbol{a_{11}} + \boldsymbol{a_{12}}$, and $\mathbf{10a_{21}} + \mathbf{a_{22}}$ are divisible by a nonzero integer $\boldsymbol{d}$, then we can write

$$\mathbf{10}\boldsymbol{a_{11}} + \boldsymbol{a_{12}} = \boldsymbol{d}(\boldsymbol{m}) \qquad (1)$$

$$\mathbf{10}\boldsymbol{a_{21}} + \boldsymbol{a_{22}} = \boldsymbol{d}(\boldsymbol{n}) \qquad (2)$$

where m and n are integers. By multiplying equation (1) by $-a_{21}$ and equation (2) by a_{11}, then adding the two equations together, we obtain:

$$-\mathbf{10}\boldsymbol{a_{11}}\boldsymbol{a_{21}} - \boldsymbol{a_{12}}\boldsymbol{a_{21}} + \mathbf{10}\boldsymbol{a_{21}}\boldsymbol{a_{11}} + \boldsymbol{a_{22}}\boldsymbol{a_{11}} = -\boldsymbol{d}(\boldsymbol{m})\boldsymbol{a_{21}} + \boldsymbol{d}(\boldsymbol{n})\boldsymbol{a_{11}}$$

It follows that

$$\boldsymbol{a_{11}}\boldsymbol{a_{22}} - \boldsymbol{a_{12}}\boldsymbol{a_{21}} = \boldsymbol{d}[(\boldsymbol{n})\boldsymbol{a_{11}} - (\boldsymbol{m})\boldsymbol{a_{21}}]$$

This shows that the determinant is divisible by d.

3 x 3 Example

Consider the square matrix B of order 3:

$$B = \begin{bmatrix} 1 & 9 & 5 \\ 3 & 1 & 5 \\ 3 & 4 & 5 \end{bmatrix}$$

In the 3 x 3 matrix above, it is notable that the concatenation of the rows yields the numbers 195, 315, and 345, all of which are divisible by 15. The determinant of $\boldsymbol{B}$ is 30, which is divisible by 15.

Theorem B. Let $\boldsymbol{B} = \begin{bmatrix} \boldsymbol{a_{11}} & \boldsymbol{a_{12}} & \boldsymbol{a_{13}} \\ \boldsymbol{a_{21}} & \boldsymbol{a_{22}} & \boldsymbol{a_{23}} \\ \boldsymbol{a_{31}} & \boldsymbol{a_{32}} & \boldsymbol{a_{33}} \end{bmatrix}$ where $\boldsymbol{a_{21}} \neq \mathbf{0}$ and $\mathbf{det}\,(\boldsymbol{B}) \neq \mathbf{0}$. If $\boldsymbol{a_{11}}\boldsymbol{a_{12}}\boldsymbol{a_{13}} = \mathbf{100}\boldsymbol{a_{11}} + \mathbf{10}\boldsymbol{a_{12}} + \boldsymbol{a_{13}}$, $\boldsymbol{a_{21}}\boldsymbol{a_{22}}\boldsymbol{a_{23}} = \mathbf{100}\boldsymbol{a_{21}} + \mathbf{10}\boldsymbol{a_{22}} + \boldsymbol{a_{23}}$, and $\boldsymbol{a_{31}}\boldsymbol{a_{32}}\boldsymbol{a_{33}} = \mathbf{100}\boldsymbol{a_{31}} + \mathbf{10}\boldsymbol{a_{32}} + \boldsymbol{a_{33}}$ are divisible by the same nonzero number $\boldsymbol{d}$, then the determinant of $\boldsymbol{B}$ is also divisible by $\boldsymbol{d}$.

Proof of theorem B: Since $\boldsymbol{a_{11}}\boldsymbol{a_{12}}\boldsymbol{a_{13}}$, $\boldsymbol{a_{21}}\boldsymbol{a_{22}}\boldsymbol{a_{23}}$, and $\boldsymbol{a_{31}}\boldsymbol{a_{32}}\boldsymbol{a_{33}}$ are divisible by the same nonzero number $\boldsymbol{d}$, then

$$\mathbf{100}\boldsymbol{a_{11}} + \mathbf{10}\boldsymbol{a_{12}} + \boldsymbol{a_{13}} = d(p_1) \qquad (3)$$

$$\mathbf{100}\boldsymbol{a_{21}} + \mathbf{10}\boldsymbol{a_{22}} + \boldsymbol{a_{23}} = d(p_2) \qquad (4)$$

$$\mathbf{100}\boldsymbol{a_{31}} + \mathbf{10}\boldsymbol{a_{32}} + \boldsymbol{a_{33}} = d(p_3) \qquad (5)$$

where p_1, p_2, and p_3 are integers. Multiplying equation (3) by $-a_{21}$ and equation (4) by $a_{11}$and adding the equations gives us:

$$10(a_{22}a_{11} - a_{12}a_{21}) + a_{23}a_{11} - a_{13}a_{21} = d[p_2a_{11} - p_1a_{21}] \tag{6}$$

Multiplying equation (4) by $-a_{31}$ and equation (5) by a_{21}, then adding the equations, we obtain:

$$10(a_{32}a_{21} - a_{22}a_{31}) + a_{33}a_{21} - a_{23}a_{31} = d[p_3a_{21} - p_2a_{31}] \tag{7}$$

Now, multiplying equation (6) by $-(a_{32}a_{21} - a_{22}a_{31})$ and equation (7) by $(a_{22}a_{11} - a_{12}a_{21})$, then adding the equations, we obtain:

$$(a_{22}a_{11} - a_{12}a_{21})(a_{33}a_{21} - a_{23}a_{31}) - (a_{32}a_{21} - a_{22}a_{31})(a_{23}a_{11} - a_{13}a_{21})$$
$$= (a_{22}a_{11} - a_{12}a_{21})\, d[p_3a_{21} - p_2a_{31}] - (a_{32}a_{21} - a_{22}a_{31})d[p_2a_{11} - p_1a_{21}]$$

Then,

$$a_{21}(a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31})$$
$$= a_{21}d[p_3a_{22}a_{11} - p_3a_{12}a_{21} + p_2a_{12}a_{31} - p_2a_{11}a_{32} + p_1a_{21}a_{32} - p_1a_{22}a_{31}]$$

Since $a_{21} \neq 0$, then we get,

$$a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$
$$= d[p_3a_{22}a_{11} - p_3a_{12}a_{21} + p_2a_{12}a_{31} - p_2a_{11}a_{32} + p_1a_{21}a_{32} - p_1a_{22}a_{31}]$$

Thus,

$$a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22}) =$$
$$= d[p_3a_{22}a_{11} - p_3a_{12}a_{21} + p_2a_{12}a_{31} - p_2a_{11}a_{32} + p_1a_{21}a_{32} - p_1a_{22}a_{31}]$$

It follows that the determinant is divisible by d.

Our aim is to generalize the methodologies elucidated for 2 x 2 and 3 x 3 matrices to square matrices of arbitrary dimensions. While it's theoretically feasible to employ the previous proofs for matrices of any size, such an approach would entail considerable time and effort.

Consequently, we will adopt a more efficient strategy by leveraging the following formula (Horn & Johnson, 1994, p. 114):

$$A^{-1} = \frac{1}{\det(A)} adj(A)$$

where A is non-singular matrix.

Theorem C. Consider

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} 10^{n-1} \\ \vdots \\ 10^{0} \end{bmatrix} = d \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix} \tag{8}$$

where the square matrix of order n is non-singular. Then, the determinant of n x n matrix is divisible by $\boldsymbol{d}$.

Proof of theorem C: Multiplying both sides of the equation (i) by A^{-1}, we have

$$\begin{bmatrix} 10^{n-1} \\ 10^{n-2} \\ \vdots \\ 10^{0} \end{bmatrix} = \frac{1}{\det(A)} adj(A)\ d \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix}$$

Again, multiplying both sides of the above equality by $\det(A)$, we get

$$\det(A) \begin{bmatrix} 10^{n-1} \\ 10^{n-2} \\ \vdots \\ 10^{0} \end{bmatrix} = adj(A)\ d \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix}$$

The last row of the equation above provides us with

$$\det(A) = adj(A) d\ p_n$$

Therefore, it can be concluded that the determinant is divisible by d.

Remark:

The converse of our theorem is not true. Here is a counter example.

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 5 \end{bmatrix}$$

The determinant of the above matrix is 5. However, 13, yielded by the concatenation of the first row, is not divisible by 5.

Corollary A:

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}^T \begin{bmatrix} 10^{n-1} \\ \vdots \\ 10^{0} \end{bmatrix} = d \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix}$$

The determinant of n x n matrix is divisible by $\boldsymbol{d}$.

Proof of Corollary A: The proof of the corollary simply follows by Zhang (2009, p. 31)

$$\boldsymbol{det(A) = det(A^T)}$$

3. Result and Discussions

The study presented herein demonstrates a notable property of square matrices, specifically focusing on the divisibility of determinants. The investigation revealed that if each row of a square matrix, where each row represents a distinct numerical value, is divisible by a common factor d, then the determinant of the matrix also exhibits divisibility by the same factor d.

The demonstration of the divisibility property of determinants in square matrices elucidates a fundamental aspect of matrix theory. By elucidating this relationship, this study contributes to the broader body of knowledge in linear algebra and provides valuable insights into the behavior of square matrices in mathematical contexts.

4. Conclusion

Matrices are important for various reasons. When the researchers study matrices on a deeper level, they can see matrices have a lot of special properties one of which has been shown in this study. In this paper, a remarkable characteristic of square matrices, namely the divisibility of determinants is illustrated with concrete examples. This study demonstrated that if a square matrix has rows that each represent a different numerical value and if these rows are divisible by a common component d, then the determinant of the matrix is also divisible by the same factor d. In the future studies, the results obtained from this study may be applied to any size square matrix and thus, matrix and graph applications may be handled by using the orthogonal matrices that have special additive determinant properties as they are worth studying in network analysis.

Declaration of Competing Interest and Ethics

The authors declare no conflict of interest. This research study complies with research publishing ethics. The scientific and legal responsibility for this manuscript published in OPS Journal belongs to the authors.

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