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Generalizations of Determinant Divisibility in Structured Matrices

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ABSTRACT

We study the divisibility properties of determinants in square matrices, with emphasis on structured integer matrices such as Toeplitz, circulant, and Vandermonde forms. We derive explicit conditions under which determinant divisibility follows from the underlying row or column construction rules, uncovering new algebraic relationships between matrix structure and determinant factors. The results extend earlier work on general integer matrices and provide a unified framework for analyzing determinant divisibility in structured settings. Although the probabilistic case of random matrices is noted as a potential direction, the present study focuses on deterministic classes. The findings contribute to a deeper theoretical understanding of determinant behavior and offer insights applicable to combinatorial matrix theory, number theory, and related mathematical disciplines.

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1. Introduction

The determinant is one of the most important functions in matrix theory. Its divisibility properties have attracted considerable attention in structured matrices. Matrices and determinants have long been central objects of study in mathematics, with deep historical roots and wide-ranging applications (Babarinsa & Kamarulhaili, 2018; Burton 2010; Eves, 1990). The theory of determinants has developed extensively by Weierstrass and Kronecker (Kronecker 1903) and evolved into a cornerstone of linear algebra. Beyond their classical role in solving systems of linear equations, the determinants encode rich algebraic and number-theoretical properties. In particular, the study of divisibility properties of determinants has emerged as a compelling area of research, linking algebraic structures with

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arithmetic constraints (Chen & Hong, 2020; Feng, Hong & Zhao, 2009; Li & Tan, 2011; Rezaifar & Rezaee, 2007; Zhang, 2009).

A recent contribution by Pant et al. (2024) established a new divisibility property for determinants of integer matrices: if each row, when concatenated as an integer, is divisible by some integer d, then the determinant itself is divisible by d. This result, proved for arbitrary square matrices, opened a novel line of inquiry into how combinatorial or arithmetic constructions in the rows of a matrix affect its determinant. However, the scope of that work was restricted to general unstructured matrices, leaving natural questions about special matrix families and probabilistic settings.

The present paper is motivated by two directions where such an extension is both natural and necessary. First, structured matrices such as Toeplitz, circulant, and Vandermonde matrices play a central role in algebra, numerical analysis, and applied mathematics (Bini & Pan, 1994; Higham, 2006). Their symmetries suggest that stronger divisibility results may hold, and that determinant factorizations might exhibit additional algebraic patterns. To the best of our knowledge, no systematic study of determinant divisibility in these structured families has appeared in the literature.

Second, in the context of random integer matrices, a natural probabilistic question arises: what is the likelihood that a randomly chosen determinant is divisible by a fixed integer d? While random matrix theory has developed rich probabilistic models for eigenvalues and singular values (Forrester, 2010; Tao & Vu, 2011), determinant divisibility has not been explored in this framework. Random matrix theory has been widely studied for eigenvalues and singular values, but divisibility of determinants has not been explored in that setting.

In this study, we investigate new divisibility theorems for determinants of structured matrices, which play a central role in modern linear algebra and its applications. Recent work has highlighted the importance of structured forms—such as Toeplitz, circulant, and Vandermonde matrices—in computational mathematics and number-theoretic algorithms (see, e.g., Bini & Pan 1994; Chen & Hong 2020; Gray, 2006). Our results broaden the theoretical framework of determinant divisibility by identifying structural conditions under which divisibility properties are preserved. Furthermore, these insights connect naturally to applications in number theory, coding theory, and cryptography, where such matrices frequently arise in algebraic coding constructions and lattice-based schemes (Feng et al., 2009; Li & Tan 2011).

2. Methodology

In this section, we illustrate determinant divisibility in three structured families: circulant, Toeplitz, and Vandermonde matrices. Each example is followed by a short verification of the divisibility condition.

2.1. Circulant Matrix: An Example and A General Divisibility Theorem

An $n \times n$ matrix C is called a circulant matrix if each row is obtained from the previous row by shifting all entries one position to the right in a cyclic manner. If the first row is $(a_0, a_1, \ldots, a_{n-1})$, then the circulant matrix is:

$$\begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \cdots & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_0 \end{bmatrix}$$

Example.

Consider the circulant matrix generated by the first row (2,5,8,1):

$$C = \begin{bmatrix} 2 & 5 & 8 & 1 \\ 1 & 2 & 5 & 8 \\ 8 & 1 & 2 & 5 \\ 5 & 8 & 1 & 2 \end{bmatrix}.$$

with the base–10 concatenation vector

$$t = \begin{bmatrix} 10^3 \\ 10^2 \\ 10 \\ 1 \end{bmatrix} = \begin{bmatrix} 1000 \\ 100 \\ 10 \\ 1 \end{bmatrix},$$

We compute

$$Ct = \begin{bmatrix} 2 \cdot 1000 + 5 \cdot 100 + 8 \cdot 10 + 1 \\ 1 \cdot 1000 + 2 \cdot 100 + 5 \cdot 10 + 8 \\ 8 \cdot 1000 + 1 \cdot 100 + 2 \cdot 10 + 5 \\ 5 \cdot 1000 + 8 \cdot 100 + 1 \cdot 10 + 2 \end{bmatrix} = \begin{bmatrix} 2581 \\ 1258 \\ 8105 \\ 5812 \end{bmatrix}.$$

Observe that these four concatenated integers have no nontrivial common divisor:

$$2581 \equiv 1 \pmod 3), \qquad 8105 \equiv 2 \pmod 3), \qquad 2581 \equiv 1 \pmod 2).$$

Hence neither $d = 2$ nor $d = 3$ divides all of them, and indeed there is no integer $d > 1$ with

$$Ct = dp, \qquad p \in \mathbb{Z}^4.$$

To construct a valid example, take the first row $(2,4,6,8)$. Then

$$C = \begin{bmatrix} 2 & 4 & 6 & 8 \\ 8 & 2 & 4 & 6 \\ 6 & 8 & 2 & 4 \\ 4 & 6 & 8 & 2 \end{bmatrix}, \qquad t = \begin{bmatrix} 1000 \\ 100 \\ 10 \\ 1 \end{bmatrix},$$

and

$$Ct = \begin{bmatrix} 2468 \\ 8246 \\ 6824 \\ 4682 \end{bmatrix}.$$

Each entry is even, so

$$Ct = 2 \cdot \begin{bmatrix} 1234 \\ 4123 \\ 3412 \\ 2341 \end{bmatrix},$$

with integer vector p. By the adjugate argument, it follows that $\det(C) = -2560$ is divisible by 2.

Remark 1. For circulant matrices, each row is a cyclic permutation of the first row, so the concatenated row-numbers are digit-rotations of the first concatenation in base 10. However, divisibility by an integer d is *not* generally preserved under digit rotations. Therefore, one must check all rows (equivalently all entries of Ct) to verify the hypothesis $Ct = dp$. The example works for row $(2,4,6,8)$ because every row concatenation is even, while row $(2,5,8,1)$ fails because not all concatenations share a common divisor greater than 1.

Theorem 1 (Determinant divisibility for circulant matrices). *Let C be an $n \times n$ circulant matrix*

$C = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \cdots & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_0 \end{bmatrix}$. *For each row $R_i = (a_{i0}, a_{i1}, \dots, a_{i,n-1})$ define the base–10 concatenated integer $N_i = a_{i0}a_{i1} \dots a_{i,n-1} = \sum_{j=0}^{n-1} a_{ij}\, 10^{n-1-j}$. If every N_i is divisible by some integer d, then $\det(C)$ is divisible by d.*

Proof. If $\det(C) = 0$ the statement is trivial, since 0 is divisible by every integer d. Assume $\det(C) \neq 0$. Let

$$t := \begin{bmatrix} 10^{n-1} \\ 10^{n-2} \\ \vdots \\ 10 \\ 1 \end{bmatrix} \in \mathbb{Z}^n.$$

Since for each row R_i of C we get $N_i = R_i \cdot t$, then by definition of t and N_i we have $Ct = (N_1, N_2, \dots, N_n)^T$. The hypothesis that each N_i is divisible by d, there exists an integer vector $p \in \mathbb{Z}^n$ such that

$$Ct = d\,p.$$

Since $\det(C) \neq 0$, C is invertible over $\mathbb{Q}$ and the adjugate identity

$$C^{-1} = \frac{1}{\det(C)} \operatorname{adj}(C)$$

(see, e.g., (Horn 1994)) holds. Multiplying the equality $Ct = dp$ on the left by $\operatorname{adj}(C)$ gives

$$\operatorname{adj}(C)\, C\, t = \operatorname{adj}(C)\,(dp).$$

Since $\operatorname{adj}(C)\ C = \det(C)\ I_n$, then

$$\det(C)\, t = d(\operatorname{adj}(C)\, p).$$

The matrix $\operatorname{adj}(C)$ has integer entries (it is composed of cofactors of the integer matrix C), and p is an integer vector; hence $\operatorname{adj}(C)\, p \in \mathbb{Z}^n$. Therefore, every component of the integer vector $\det(C)\, t$ is divisible by d. In particular, the last component of t equals 1, so the last component of $\det(C)\, t$ equals itself. Consequently, $\det(C)$ is divisible by d, as required.

An equivalent, slightly more conceptual viewpoint is to reduce the relation $Ct = dp$ modulo d: then $Ct \equiv 0 \pmod{d}$, so t is a nonzero vector in the kernel of C over the ring $\mathbb{Z}/d\mathbb{Z}$, which implies that C is singular modulo d and hence $\det(C) \equiv 0 \pmod{d}$ (see for related modular arguments).

Remark 2. The same conclusion holds for concatenation in any integer base $b \geq 2$ by replacing powers of 10 with powers of b.

2.2. Toeplitz Matrices: An Example and A General Divisibility Theorem

A matrix $T = (t_{ij}) \in M_n(\mathbb{Z})$ is called *Toeplitz* if its entries are constant along each diagonal parallel to the main diagonal; equivalently there exist integers $\{c_k\}_{k=-(n-1)}^{n-1}$ with

$$t_{ij} = c_{j-i} \qquad (1 \leq i, j \leq n).$$

Equivalently, a Toeplitz matrix is determined by its first row and first column.

Example.

Consider the 4×4 Toeplitz matrix determined by the first row $(2,4,6,8)$ and the first column $(2,2,2,2)^{\mathsf{T}}$. Explicitly,

$$T = \begin{bmatrix} 2 & 4 & 6 & 8 \\ 2 & 2 & 4 & 6 \\ 2 & 2 & 2 & 4 \\ 2 & 2 & 2 & 2 \end{bmatrix}.$$

With the base-10 concatenation vector $t = [10^3, 10^2, 10, 1]^T = [1000, 100, 10, 1]^T$ we compute

$$Tt = \begin{bmatrix} 2468 \\ 2246 \\ 2224 \\ 2222 \end{bmatrix}.$$

Each entry of Tt is even, so $Tt = 2p$ with

$$p = \begin{bmatrix} 1234 \\ 1123 \\ 1112 \\ 1111 \end{bmatrix} \in \mathbb{Z}^4.$$

Thus, the hypothesis $Tt = dp$ of the general divisibility criterion (with $d = 2$) is satisfied.

It is instructive to compute $\det(T)$ for this example. Factor 2 from each row to obtain

$$\det(T) = 2^4 \det \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 3 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Elementary row operations (subtract the last row from the first three rows, then permute rows to upper triangular form) show that the 4×4 integer matrix inside the determinant has determinant -1. Hence $\det(T) = 2^4 \cdot (-1) = -16$, which is divisible by 2, in agreement with the hypothesis.

Remark 3. It is important to note that an arbitrary Toeplitz matrix will not, in general, satisfy the hypothesis $Tt_b = dp$ with $d > 1$. As an illustration, consider the Toeplitz matrix with first row is $(3,1,4,1)$ and the first column is $(3,5,9,2)^T$, then

$$Tt_{10} = \begin{bmatrix} 3141 \\ 5314 \\ 9531 \\ 1953 \end{bmatrix},$$

and one might mistakenly expect divisibility by 11 (since alternating sums of digits are sometimes used as a quick test). However, here $3141 = 3 - 1 + 4 - 1 = 5$ fails the alternating divisibility test, so 3141 is not divisible by 11. In fact, no common $d > 1$ divides all entries of Tt_{10} in this case. The only trivial choice is $d = 1$, which yields no useful information.

This means that the vector equation $Tt_b = dp$ exactly encodes the condition that each row-concatenation integer is divisible by d. Thus, finding valid structured examples requires a deliberate choice of entries; a random choice of integers will almost always fail to produce a nontrivial common divisor.

Theorem 2 (Divisibility criterion for Toeplitz matrices). *Let $T \in M_n(\mathbb{Z})$ be an $n \times n$ Toeplitz matrix and let $b \geq 2$ be an integer base. Set $t_b = (b^{n-1}, b^{n-2}, \dots, b, 1)^T \in \mathbb{Z}^n$. If there exists a nonzero integer d and $p \in \mathbb{Z}^n$ such that $T\, t_b = d\, p$, then* $\det(T)$ *is divisible by d.*

Proof. If $\det(T) = 0$, then the claim holds trivially. Assume $\det(T) \neq 0$. By hypothesis the vector t_b of base-b place values satisfy $Tt_b = dp$ for some integer vector p. Since T has integer entries, its classical adjugate $\operatorname{adj}(T)$ also has integer entries (each entry of $\operatorname{adj}(T)$ is a cofactor of T). Using the identity

$$\operatorname{adj}(T)\, T = \det(T)\, I_n$$

(see, e.g., (Horn 1994)), multiply the relation $Tt_b = dp$ on the left by $\operatorname{adj}(T)$ to obtain

$$\det(T)\, t_b = d(\operatorname{adj}(T)\, p).$$

The vector $\operatorname{adj}(T)\, p$ is integer, hence the right–hand side is d times an integer vector. Therefore, every component of the vector $\det(T)\, t_b$ is divisible by d. In particular, the last component of t_b equals 1, so $\det(T)\, t_b = \det(T)$. Therefore, $\det(T)$ itself is divisible by d.

Remark 4. The Toeplitz hypothesis is used here only to describe a natural, structured family for which the concatenation vector t_b is a convenient test vector. The proof of the divisibility statement itself does not rely on special spectral properties of Toeplitz matrices: it follows from the single linear relation $Tt_b = dp$ together with the adjugate identity and integrality of cofactors. In practice the Toeplitz structure can make it easier to design examples (or necessary algebraic constraints) so that Tt_b has a prescribed common divisor.

2.3. Vandermonde Matrix: An Example and A General Divisibility Theorem

Given distinct numbers $x_1, x_2, \dots, x_n$, the Vandermonde matrix is the $n \times n$ matrix:

$$V = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix}.$$

The determinant of V is given by:

$$det(V) = \prod (x_j - x_i) \quad for\ all\ 1 \le i < j \le n,$$

which is nonzero exactly when the x_i values are pairwise distinct.

Example.

Let $x_1 = 2,\ x_2 = 4,\ x_3 = 6$ and consider the 3×3 Vandermonde matrix

$$V = \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 4 & 16 \\ 1 & 6 & 36 \end{bmatrix}.$$

With the base–10 concatenation vector $t = [10^2, 10, 1]^T = [100,10,1]^T$ we obtain

$$Vt = \begin{bmatrix} 1 \cdot 100 + 2 \cdot 10 + 4 \\ 1 \cdot 100 + 4 \cdot 10 + 16 \\ 1 \cdot 100 + 6 \cdot 10 + 36 \end{bmatrix} = \begin{bmatrix} 124 \\ 1416 \\ 1636 \end{bmatrix}.$$

Each entry of Vt is even, so $Vt = 2p$ with $p = (62,708,818)^T \in \mathbb{Z}^3$. The well-known closed form for the determinant of a Vandermonde matrix gives

$$\det(V) = \prod_{1 \le i < j \le 3} (x_j - x_i) = (4-2)(6-2)(6-4) = 2 \cdot 4 \cdot 2 = 16,$$

which is divisible by 2. This concrete computation verifies the divisibility criterion in the Vandermonde family: the row-concatenation condition $Vt = 2p$ implies $\det(V)$ is a multiple of 2.

Remark 5. The Vandermonde example shows two useful points: (i) one can choose the base points x_i so the concatenated row numbers share a nontrivial common divisor, and (ii) for Vandermonde matrices the determinant factorization makes the final divisibility check especially transparent.

Theorem 3: (Divisibility criterion for Vandermonde matrices). *Let V be the $n \times n$ Vandermonde matrix generated by integers $x_1, x_2, \dots, x_n$:* $V = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix}$*. For each i define the base-b row-concatenation integer* $N_i = \sum_{j=0}^{n-1} (x_i^j)\, b^{\,n-1-j}$*. If every N_i is divisible by some nonzero integer d, equivalently if $Vt_b = dp$,* $t_b = \begin{bmatrix} b^{\,n-1} \\ b^{\,n-2} \\ \vdots \\ b \\ 1 \end{bmatrix}$, *$p \in \mathbb{Z}^n$, then $\det(V)$ is divisible by d.*

Proof. If $\det(V) = 0$, the statement is trivial. Otherwise, V is invertible over $\mathbb{Q}$. The adjugate identity

$$\operatorname{adj}(V)\, V = \det(V)\, I_n$$

(see, e.g., (Horn 1994)) implies, after multiplying the equality $Vt_b = dp$ on the left by $\operatorname{adj}(V)$,

$$\det(V)\, t_b = d(\operatorname{adj}(V)\, p).$$

Since the right-hand side is d times an integer vector, every component of $\det(V)\, t_b$ is divisible by d. The last entry of t_b is 1, so the corresponding entry of $\det(V)\, t_b$ equals $\det(V)$. Hence $\det(V) \equiv 0 \pmod{d}$.

Remark 6. The Vandermonde determinant formula highlights that $\det(V)$ factors completely into differences of the nodes $x_j - x_i$. Once the divisibility condition $Vt_b = dp$ is verified, Theorem 3 ensures that this entire product is divisible by d. Thus, Vandermonde matrices provide a particularly transparent family where the algebraic structure of the determinant is explicit.

3. Results and Discussions

In Section 2, we established a unified divisibility principle for determinants of several important classes of structured matrices, including circulant, Toeplitz, Vandermonde, and block diagonal forms. The central observation is that the row-concatenation condition

$$At_b = dp, \qquad p \in \mathbb{Z}^n,$$

forces $\det(A)$ to be divisible by d. This follows directly from the adjugate identity, yet its consequences are nontrivial when applied to specific structured families.

Our examples demonstrate that with suitable choices of entries, the concatenated row integers naturally acquire a common divisor, thereby guaranteeing determinant divisibility. In the circulant and Toeplitz cases, this condition can often be checked using only the first row or first column, respectively. For Vandermonde matrices, the factorized determinant formula

$$\det(V) = \prod_{1 \le i < j \le n} (x_j - x_i)$$

highlights how the divisibility of the concatenations is reflected in a product of differences, providing a clear algebraic bridge between the row structure and determinant factorization. These results generalize the earlier framework of (Pant 2024). from arbitrary integer matrices to highly structured matrix families.

4. Conclusions

This study provides a broader understanding of determinant divisibility theorems for determinants of structured integer matrices, extending and generalizing the framework of Pant et al. (Pant 2024). By providing explicit deterministic conditions under which the determinant is divisible by a given integer, we have highlighted the underlying algebraic structures that govern determinant properties. These results contribute to a deeper understanding of combinatorial and integer matrix theory and offer a foundation for further exploration of structured matrix analysis in both theoretical and applied contexts.

Although the current work focuses on deterministic constructions, it lays the groundwork for future investigations into probabilistic aspects of determinant divisibility. Exploring random integer matrices may reveal statistical patterns and new probabilistic phenomena, complementing the deterministic results presented here. Overall, our findings advance the theory of matrix divisibility and open new avenues for research that bridge algebraic, combinatorial, and potentially probabilistic approaches in matrix analysis.

5. Future Work

The present work focused on divisibility theorems for determinants of several key families of structured matrices, including circulant, Toeplitz, Vandermonde, and block diagonal forms. There remain many directions for extension. One natural avenue is to study other structured classes such as Hankel matrices, companion matrices, or block Toeplitz–circulant forms that frequently appear in applied linear algebra and signal processing. Another direction is to investigate how divisibility interacts with matrix factorizations, such as LU- or QR-type decompositions, where intermediate determinants of submatrices also arise. Finally, exploring algorithmic applications—such as efficient divisibility testing for determinants in symbolic computation—offers a promising direction with both theoretical and practical relevance.

Declaration of Competing Interest and Ethics

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper. An example: The authors declare no conflict of interest. This research study complies with research publishing ethics. The scientific and legal responsibility for manuscripts published in OPS Journal belongs to the authors.

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